ON CHARACTERIZATIONS OF LINEAR GROUPS, I[1]

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Let L_q be the totality of nonsingular 2×2 matrices with determinant 1 over a finite field of q elements. This paper is concerned with a group-theoretical characterization of L_q in the case that q is a power of 2. We shall call an element of order 2 an *involution*. It is easy to see that L_q $(q=2^n)$ contains an involution τ such that the centralizer of τ in L_q is an abelian 2-group. In his thesis [5] K. A. Fowler proved that this property (together with nonsolvability) actually characterizes L_q. A few years later working independently G. E. Wall and the author obtained a generalization of Fowler's theorem characterizing L_q under weaker assumptions (cf. [3]). The purpose of this paper is to prove a further generalization of these results: i.e. to prove the following theorem.

MAIN THEOREM. *Let G be a finite group of even order. If the centralizer of any involution in G is always abelian then we have one of the following three possibilities*: (1) *2-Sylow subgroups of G are cyclic*, (2) *a 2-Sylow subgroup of G is a normal subgroup*, or (3) *G is a direct product of two groups L and A where L is one of the linear groups L_q with $q=2^n$ and A is an abelian group of odd order.*

The proof of this result is rather complicated. The first section is devoted to a reduction process. Assuming our main theorem to be false we study the structure of the group with the smallest order for which the assertion of the main theorem is not true. The second section is concerned with a particular type of finite groups with structure similar to the one obtained at the end of the first section. In view of further applications we take slightly more general assumptions. Our main purpose is to obtain a formula for the order of such a group which is applied in the next section to prove the nonexistence of the group considered in §1. The method used here is similar to the one in [1] or [6]. An application of our main theorem will be given in the final section.

1. We use the following notations. Let G be a finite group, S a 2-Sylow subgroup, Z the centralizer of S and N the normalizer of S in G. Throughout this section we assume that

(1) *The centralizer of any involution in G is abelian.*

Under this assumption we have the following proposition.

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PROPOSITION 1. *The 2-Sylow subgroup S is abelian, Z is a direct product of S and another abelian group A, and Z is the centralizer of each involution in S.*

Proof. From a property of 2-groups S contains an involution τ contained in the center of S. Then the centralizer of τ contains S and Z, and is abelian by the assumption (1). Hence both S and Z are abelian. We have thus shown the first two assertions. If τ' is another involution of S, the centralizer of τ' contains Z and is again abelian. Hence it must coincide with Z.

From now on we shall assume moreover that

(2) *A 2-Sylow subgroup of G is neither normal nor cyclic.*

We shall use induction on the order of G in order to prove our main theorem so that we assume the validity of our theorem for groups with smaller order than G.

PROPOSITION 2. *If G contains a normal subgroup $P\neq e$ of odd prime power order, then G is a direct product of L_q and an abelian group A of odd order.*

Proof. Let $\overline{G}=G/P$. By assumption $\overline{G}$ has a smaller order than G. Let τ' be an involution of $\overline{G}$. Since P is a group of odd order we may take an involution τ of G contained in the coset τ'. The centralizer of τ' in $\overline{G}$ has the form U/P with a subgroup U of G. Since the subgroup $\{\tau\}$ generated by τ is a 2-Sylow subgroup of $\{P, \tau\}$ and $\{P, \tau\}$ is a normal subgroup of U, we conclude that U is the join of $\{P, \tau\}$ and the centralizer V of τ in U. Then $U=P\cup V$ and $U/P\cong V/P\cap V$. V is by the assumption (1) an abelian group and so is $V/P\cap V$. Hence U/P is abelian. Since τ' was any involution of $\overline{G}$, $\overline{G}$ satisfies the condition (1).

If S is a 2-Sylow subgroup of G, $\overline{S}=SP/P$ is a 2-Sylow subgroup of $\overline{G}$ and is isomorphic with S. Hence $\overline{S}$ is not cyclic by (2). Suppose that $\overline{S}$ is a normal subgroup of $\overline{G}$. S is not a normal subgroup of PS, if so S would be a normal subgroup of G contradicting (2). If $S'\neq S$ is a conjugate subgroup of S in SP, we have $S\cap S'=e$, otherwise $S\cap S'$ contains an involution, of which the centralizer is not abelian. If $\tau\in S$ and $\tau'\in S'$ are both involutions, τ and τ' are conjugate in SP. Hence all involutions in S are conjugate. Burnside's argument yields that they are conjugate in the normalizer of S, which is impossible since the normalizer of S in SP is a direct product of S and a subgroup of P. Hence $\overline{S}$ is not a normal subgroup of $\overline{G}$. $\overline{G}$ satisfies therefore the conditions (1) and (2) and has smaller order than G.

Applying the inductive hypothesis to $\overline{G}$ we conclude that $\overline{G}=\overline{L}\times\overline{A}$ where $\overline{L}$ is a linear group L_q and $\overline{A}$ is an abelian group of odd order. Let $\overline{L}=L/P$ and $\overline{A}=A/P$. Then A is of odd order. If W/P is a 2-Sylow subgroup of L/P, W is a direct product of P and a 2-Sylow subgroup S of G. This fact may be proved in the same way as before. Hence P is in the center of W. Since W/P may be any 2-Sylow subgroup of $\overline{L}$, P is in the center of L. We have assumed that P is of odd prime power order and so every Sylow subgroup of L is abelian since every Sylow group of odd order in $\overline{L}$ is cyclic. If the order of

P is relatively prime to the order of L/P, applying Schur's splitting theorem [9, p. 125] we see that L is a direct product of P and a group L_0 isomorphic with L/P. If the order of P is not prime to that of L/P, we may apply Theorem 14 of [8] to show the validity of the same conclusion. In any case L_0 is a normal subgroup of G and $G=L_0\times A$. Since A is a part of the centralizer of an involution, A is an abelian group.

PROPOSITION 3. *If G contains a proper normal subgroup G_0 which contains S, then the same conclusion as in Proposition 2 holds.*

Proof. Clearly G_0 satisfies the assumptions (1) and (2) and has smaller order than G. Hence the inductive hypothesis yields that G_0 is a direct product of a linear group L and an abelian group A_0. L is a normal subgroup of G as being a characteristic subgroup of G_0. G may be represented as a permutation group Γ on the 2-Sylow subgroups of G. Since L contains a 2-Sylow subgroup S of G, G is the join of L and the normalizer N of S. $N'=N\cap L$ is the normalizer of S in L. Hence the degree of Γ is the index $[L: N']$, which we denote by $1+q$. Γ is clearly triply transitive. If σ is an element which leaves three 2-Sylow subgroups fixed, σ commutes with at least one involution of each 2-Sylow subgroup left fixed by σ. Hence σ is in the centralizer of L. Denoting by A the centralizer of L we conclude that Γ is isomorphic with G/A and by a theorem of Zassenhaus [7] G/A is isomorphic with the linear group L_q. Hence comparing the orders we see that $G=L\times A$ as claimed.

In the proof of our main theorem we need a generalized form of Fowler's theorem obtained by G. E. Wall and the author. This result may be stated in the following way. Let G be a finite group of even order satisfying the conditions (1) and (2). Let S be a 2-Sylow subgroup of G, Z the centralizer and N the normalizer of S in G. By Proposition 1, Z is an abelian group and $Z=S\times A$. *If we assume moreover that the centralizer of any element $\neq 1$ of A is contained in N then $G=L_q$ with $q=2^n=[S: e]$.* The particular case of $A=e$ is the theorem of Fowler [5]. This generalization is given in [3].

We return to our case. By way of contradiction we assume that G is not of the form of a direct product $L\times A$. Then from the above result we conclude that if the centralizer Z of a 2-Sylow subgroup S is of the form $S\times A$, A contains an element $\sigma_0\neq 1$ such that the centralizer of σ_0 is not a part of N, the normalizer of S in G. If σ_0 is in the center of G, we conclude that $G=L\times A$ by Proposition 2. Hence σ_0 is not in the center; i.e. the centralizer of σ_0 is a proper subgroup of G. Let M be a maximal subgroup of G which contains the centralizer of σ_0 in G. From the choice of σ_0 it follows that the 2-Sylow subgroups of M are not self-conjugate, whence we may apply the inductive hypothesis to M. M is then the direct product of L and A. If σ is any element $\neq 1$ of A, the centralizer of σ contains M. If this centralizer is not M, σ is in the center of G since M is by definition a maximal subgroup of G. Hence G itself is a direct product of a linear group and an abelian group by Proposition

2. This contradicts the assumption. Hence M is the centralizer of any element $\neq 1$ of A.

Let A_p be a p-Sylow subgroup $\neq e$ of A. Then A_p is a normal subgroup of M. If A_p is a normal subgroup of G we may apply the same argument as above to derive a contradiction. Hence the normalizer of A_p in G is the subgroup M. If A_p is a p-Sylow subgroup of G, A_p is in the center of its normalizer as the latter is M. Hence by a theorem of Burnside [4, §243] G contains a normal subgroup G_0 such that $G=G_0A_p$ and $A_p\cap G_0=e$. Since $G/G_0\cong A_p$ is of odd order we have $G_0\supseteq S$. Applying Proposition 3 we see that this gives a similar contradiction. Let G_p be a p-Sylow subgroup of G containing A_p. G_p contains a subgroup U such that $[U: A_p]=p$. A_p is a normal subgroup of U and hence U is a part of M. Hence p divides the order of L. Let L_p be a p-Sylow subgroup of L. From the structure of L, L_p is a cyclic group. Since p divides the order of L, $L_p\neq e$. $L_p\cup A_p=V$ is a p-Sylow subgroup of M. From the assumption we see that the centralizer W of V in G is a subgroup of M. The normalizer H of V contains W as a normal subgroup. If H is also a part of M, A_p is in the center of H. In this case V is a p-Sylow subgroup of G and hence applying a theorem of Zassenhaus (loc. cit. [8, Theorem 14]) we conclude that G contains a normal subgroup G_0 such that $G/G_0\cong A_p$. This is however impossible in virtue of Proposition 3. Hence we see that H is not a part of M and contains an element $\rho\notin M$. Then $\rho A_p\rho^{-1}$ is a subgroup of V and different from A_p. Any element of $A_p\cap\rho A_p\rho^{-1}$ commutes elementwise with M and $\rho M\rho^{-1}\neq M$. Hence we conclude that $A_p\cap\rho A_p\rho^{-1}=e$. This implies that $A_p\cong\rho A_p\rho^{-1}\cong(\rho A_p\rho^{-1})A_p/A_p\subseteq V/A_p\cong L_p$. Hence A_p is a cyclic group of order a divisor of $[L_p: e]$. Since p is a divisor of the order of L, p is a divisor of $q+\delta$ ($\delta=\pm 1$, $q=[S: e]$). If the order of A is divisible by a prime number r which divides $q-\delta$, A_r is a characteristic subgroup of W. Hence A_r is a normal subgroup of H, which is a contradiction since $M\not\supseteq H$. Hence A is a cyclic group of order a divisor of $q+\delta$. The discussion may be summarized as follows.

Proposition 4. *If the conclusion of the main theorem is not true, then there exists a group G which possesses the following properties*: (1) *a 2-Sylow subgroup S is neither cyclic nor normal*, (2) *the centralizer of any involution in S is abelian and coincides with the centralizer Z of S and $Z=S\times A$*, (3) *the centralizer M of an element $\neq 1$ of A is a direct product $L_q\times A$, is the centralizer of every element $\neq 1$ of A, and is a non-normal maximal subgroup and* (4) *A is a cyclic group of order a divisor of $q+\delta$* ($\delta=\pm 1$).

2. In this section we shall consider a finite group G which satisfies the following properties:

(A) G contains an abelian subgroup Z of order $h>1$ and a subgroup M such that M is the normalizer of any subgroup $\neq e$ of Z, $M=L_q\times Z$ where $L_q\cong LF(2, q)$, $q=2^\mu>2$ and h is a divisor of $q+\delta$ ($\delta=\pm 1$) and

(B) if τ is an involution of G, the centralizer N of τ contains a normal subgroup N_0 such that N_0 is a 2-group and $N/N_0 \cong Z$.

Later we shall make an additional assumption. The purpose of this section is to obtain a formula for the order of G.

The order of L_q is $q(q-1)(q+1)$. From the structure of L_q we see that L_q contains a cyclic subgroup U of order $q-\delta$. Let H be the normalizer of any subgroup $U_0 \neq e$ of U in G. The centralizer H_0 of U_0 is a normal subgroup of H and the factor group H/H_0 is abelian as being isomorphic with a subgroup of the automorphism group of a cyclic group. H contains Z. If $Z_p \neq e$ is a p-Sylow subgroup of Z the normalizer V of Z_p in H is the intersection of M and H. Hence V is the nornalizer of U_0 in M and therefore $V = D \times Z$ where D is a dihedral subgroup of L_q of order $2(q-\delta)$. This implies that Z_p is a p-Sylow subgroup of H and is in the center of its normalizer V. By a theorem of Burnside (loc. cit.) H contains a normal subgroup H_p such that $H = H_p Z_p$ and $H_p \cap Z_p = e$. This is true for every prime divisor of the order of Z and hence H contains a normal subgroup K such that $KZ = H$, $K \cap Z = e$ and the order of K is relatively prime to the order h of Z. Let $W = K \cap H_0$. Then W is a normal subgroup of H and the order of W is relatively prime to h. Since U_0 is in the center of W every element of W commutes with any element of U_0. By the second assumption (B) any element $\neq 1$ of U_0 does not commute with any involution. Hence we conclude that the order of W is odd. From the structure of L_q there is an involution τ of L_q which maps every element of U_0 into its inverse. This element τ is in H and hence transforms W into itself. Since the order of W is relatively prime to $2h$, τ leaves only the identity of W invariant. Hence τ maps every element of W into its inverse and W is abelian.

Consider the normalizer X of W in G. In X the normalizer of H_0 is the join of H_0 and the normalizer of Z in X (cf. [9, p. 126, Theorem 26]). The normalizer of Z in X is by (A) the intersection of M and X which is the subgroup of M consisting of elements transforming W into itself. If $\sigma \in M \cap X$, then $\sigma W \sigma^{-1} = W$ and hence $\sigma(M \cap W)\sigma^{-1} = M \cap W$. On the other hand W is an abelian subgroup of order relatively prime to $2h$. Hence $L_q \supseteq M \cap W \supseteq U$, which implies that $M \cap W = U$. This means that σ is in the normalizer of U in M. Hence $M \cap X = V$. Thus the normalizer of H_0 in X is $H_0 \cup V \subseteq H$. Since $X \supseteq H$ this normalizer contains H. Hence $H = H_0 \cup V$ and H is the normalizer of H_0 in X. Moreover we conclude that $H/H_0 \cong V/H_0 \cap V$ has order 2: $[H: H_0] = 2$.

From the above result and the fact that W is abelian we may prove that H and H_0 are independent of the choice of U_0: i.e. H is the normalizer and H_0 is the centralizer of any subgroup $U_0 \neq e$ of U. Suppose that $U_0 = U$ was our first choice. Then W is an abelian group containing U and $[H: W] = 2h$. If U_0' is arbitrary subgroup $\neq e$ of U, we have corresponding subgroups H' and W'. Since $W \supseteq U \supseteq U_0$, we have $W' \supseteq W$. On the other hand W' is again an abelian group containing U. Hence $W \supseteq W'$. Thus $W = W'$ and hence $H = H'$.

We shall next prove that the normalizer Y of any subgroup Y_0 between

H_0 and W which is not equal to W is H. Use induction on the index $[H_0: Y_0]$. Applying the inductive hypothesis and a theorem of Burnside (loc. cit.) to the factor group Y/Y_0 we conclude that Y contains a normal subgroup Y_1 such that $Y_1 \cap H_0 = Y_0$ and $Y_1 \cup H_0 = Y$. Using the same argument as before we see that Y_1 is the join of Y_0 and the normalizer of $Y_0 \cap Z$ in Y_1. Hence again $[Y_1: Y_0]=2$ and $Y_1 \subseteq H$. Therefore $Y=H$.

Burnside's theorem applied to X/W shows that X contains a normal subgroup X_0 such that $X_0 \cup H_0 = X$ and $X_0 \cap H_0 = W$. If τ and τ' are two involutions of X_0, both τ and τ' map every element of W into its inverse. Hence $\tau^{-1}\tau'$ belongs to the centralizer H_0 of U_0. Hence τ and τ' are in the same coset modulo W and so X_0/W contains only one involution. Hence X is the join of W and the centralizer of τ in X. By the isomorphism theorem and by the second assumption (B) we conclude that X_0/W is a 2-group. Since τW is the only involution in X_0/W, X_0/W is either cyclic or a generalized quaternion group. We have shown in the preceding paragraph that the centralizer of any proper subgroup of H_0/W in X/W is H/W. Hence we see that either $X_0 = K$ and $X=H$ or else X_0/W is the quaternion group of order 8 and $h=3$. We shall now introduce another assumption on G.

(C) If the centralizer of some involution is not abelian, then $h=q+\delta$.

Using this assumption we can eliminate the second possibility. By way of contradiction suppose $X \neq H$. Then we have shown that X_0/W is the quaternion group of order 8 and is isomorphic with a subgroup of the centralizer of τ. By assumption (C) we have $h=q+\delta$, since the centralizer of τ is not abelian. Since $h=3$ in this case and we have assumed that $q>2$, we conclude that $q=4$ and $\delta=-1$. The order of U is therefore $q-\delta=5$. U has exactly $[X:H]=4$ conjugate subgroups in X. These conjugate subgroups of U generate a subgroup U_1 of order 5^λ ($\lambda \leqq 4$). The groups X_0/W and H_0/W induce automorphisms of U_1 and U_1/U respectively and the identity is the only invariant element in each case. Hence we have simultaneous congruences $[U_1: e] \equiv 1 \pmod 8$ and $[U_1: U] \equiv 1 \pmod 3$, or $5^\lambda \equiv 1 \pmod 8$ and $5^{\lambda-1} \equiv 1 \pmod 3$. These simultaneous congruences are impossible. This contradiction shows the validity of our assertion.

We shall next show that if σ is an element of W not contained in U then the centralizer R of σ is W. Again by way of contradiction suppose $R \neq W$. If R is abelian, R is contained in H_0 as $R \supseteq W \supseteq U$. Hence $R = W \times (R \cap Z)$ and $R \subseteq M \cap H_0$, since $R \cap Z \neq e$. On the other hand $M \cap H_0 = U \times Z$ is abelian. This implies that $W=U$ contradicting the assumption $\sigma \notin U$. Hence R is not abelian. If τ is an involution of H which commutes with every element of Z, we have $\tau\sigma\tau^{-1} = \sigma^{-1}$ and hence τ maps R into itself. There must be an element $\rho \neq 1$ of R which commutes with τ, otherwise R would be abelian. The order of σ is relatively prime to $2h$, and σ does not commute with any involution by the assumption (B). This implies in particular that the order of R is odd. Again by (B) ρ is conjugate to an element of Z in the centralizer of τ. If the centralizer of τ is abelian ρ itself is an element of Z. Hence $\sigma \in M \cap W = U$. This is not the case since we have assumed $\sigma \notin U$. If the centralizer of τ is not abelian the assumption (C) on G yields $h = q + \delta$. As shown above a conjugate element $\pi\rho\pi^{-1}$ ($\pi \in G$) is in Z and commutes with $\pi\sigma\pi^{-1}$. Hence $\pi\sigma\pi^{-1}$ is an element of M. Since $\sigma \in W$, $\pi\sigma\pi^{-1}$ has an order relatively prime to $2h = 2(q+\delta)$. Therefore $\pi\sigma\pi^{-1}$ is an element of L_q with an order dividing $q - \delta$. From the structure of L_q it follows that $\pi\sigma\pi^{-1}$ is conjugate to an element of U. This means that σ is conjugate to an element of U in G and hence the centralizer R of σ is isomorphic with H_0. In particular R contains a normal subgroup of index h. Since $W \subseteq R$ we conclude that W is a normal subgroup of R. Thus R is a part of the normalizer of W, the latter being H. Since H_0 is the only subgroup of H of index 2 we see that $R = H_0$. There is an element π' of G such that $\pi'\sigma\pi'^{-1} \in U$. Since R is the centralizer of σ, $\pi' R \pi'^{-1}$ is the centralizer of $\pi'\sigma\pi'^{-1}$. Hence $\pi' R \pi'^{-1} = H_0$. Since $R = H_0$, we see that π' is in the normalizer of H_0. Hence $\pi' \in H$. This is impossible because $\sigma \in \pi'^{-1} U \pi' = U$ while $\sigma \notin U$ by assumption.

PROPOSITION 5. *Let G be a group satisfying conditions* (A), (B) *and* (C). *Let U be a subgroup of M with order $q-\delta$ and let H be the normalizer of U in G. Then* (i) *H contains an abelian normal subgroup W of index $2h$ and of order relatively prime to $2h$,* (ii) *H is the normalizer of W in G as well as the normalizer of any subgroup $\neq e$ of U,* (iii) *the centralizer of any element $\neq 1$ of U is a fixed subgroup H_0 of H with index 2, while the centralizer of any element of $W - U$ is W. Moreover* (iv) *H contains a normal subgroup H_1 of index $2(q-\delta)$ such that W is the direct product of U and $W \cap H_1$.*

The first part of this proposition is just a summary of the results so far obtained. The last assertion may be proved by using the following lemma: Let Γ be a finite group of operators of a finite abelian group A and let B be the totality of elements of A which are left invariant by Γ. If the order of Γ is relatively prime to the order of A, then there exists a Γ-invariant subgroup C of A such that $A = B \times C$. (Apply this lemma to the Z-group W.) The existence of H_1 may also be proved directly by applying a theorem of Zassenhaus [8, Theorem 14] to H_0.

We shall use the character theory and apply the method of [1] and [6]. There is an involution τ in H which commutes with every element of Z. Let C be the totality of elements of H conjugate to some element of $\{Z, \tau\}$ and C' be the rest of the elements in H. If $\sigma \in C'$, the centralizer of σ is contained in H. If two elements σ and σ' of C' are conjugate in G, then they are conjugate in H. In fact if $\rho\sigma\rho^{-1} = \sigma'$, ρ maps the centralizer of σ into that of σ'. These centralizers are by (iii) of Proposition 5 $U \times Z$, W or H_0. In any case ρ maps W (or U) into itself and hence, by (ii) of Proposition 5, $\rho \in H$. Thus every conjugate class of H in C' is special (cf. [6, Lemmas 4,5]).

All the irreducible characters of H may be obtained without much difficulty. Let $[W: U]=m$ and $q-\delta=2t+1$. Let ω_k $(k=1, 2, \cdots, h)$ be the totality of linear characters of Z. Then H has exactly $2h$ linear characters $\xi_1, \cdots, \xi_h$ and $\eta_1, \cdots, \eta_h$. The values of these characters may be computed as follows:

$$\begin{aligned} \xi_i(\pi) &= \eta_i(\pi) = 1 \qquad \text{if } \pi \in W, \\ \xi_i(\rho\sigma) &= \xi_i(\tau\sigma) = \eta_i(\rho\sigma) = -\eta_i(\tau\sigma), \\ &= \omega_i(\sigma) \qquad \text{if } \sigma \in Z,\ \rho \in U. \end{aligned}$$

By (iv) of Proposition 5, H contains a normal subgroup H_1 of index $2(q-\delta)$ and the factor group H/H_1 is a dihedral group isomorphic with $\{U, \tau\}$. The group $\{U, \tau\}$ has t characters $\phi_1, \cdots, \phi_t$ of degree 2. H has exactly th characters ϕ_{ij} $(i=1, 2, \cdots, t; j=1, 2, \cdots, h)$ of degree 2. The values of these characters are:

$$\begin{aligned} \phi_{ij}(\pi\rho) &= \phi_i(\rho) \qquad \text{if } \rho \in U \text{ and } \pi \in W \cap H_1, \\ \phi_{ij}(\sigma\rho) &= \phi_i(\rho)\omega_j(\sigma) \qquad \text{if } \sigma \in Z \text{ and } \rho \in U, \\ \phi_{ij}(\tau\sigma) &= 0 \qquad \text{if } \sigma \in Z. \end{aligned}$$

H has $s=(m-1)(q-\delta)/2h$ more characters $\theta_1, \cdots, \theta_s$ of degree $2h$. These characters are the induced characters from linear characters of W with kernels not containing $W\cap H_1$.

From the character table we can construct linear combinations of irreducible characters with integral coefficients vanishing on C. We see that

$$(*) \qquad \phi_{ij}-\xi_j-\eta_j, \qquad \phi_{ik}-\phi_{jk}, \qquad \theta_i-\theta_j \quad \text{and} \quad \theta_i-\sum\nolimits_k\phi_{ik}$$

vanish on C. From the orthogonality relations we conclude that if a linear combination ψ of irreducible characters with integral coefficients is orthogonal to all the characters in (*) except the last type then ψ has the form $a\sum_i \theta_i + \sum_k a_k \sum_i \phi_{ik} + \sum_k (x_k\xi_k+y_k\eta_k)$ with $x_k+y_k=a_k$. Hence ψ vanishes on the classes containing elements $\rho\sigma$ with $1\neq\rho\in U$ and $\sigma\in Z$. If we assume moreover that ψ is orthogonal to the last type of characters too, then we have an additional relation $a=\sum_k a_k$. Hence ψ vanishes on the elements $\pi\neq 1$ of W as well as $\rho\sigma$. It follows that the value of ψ at the identity is divisible by $m(q-\delta)$.

Since the classes in C' are special we may apply Lemmas 4 and 5 of [6] to the characters (*). Let $\psi_{ij}=\phi_{ij}-\xi_j-\eta_j$. By Lemma 4 of [6] the induced character ψ_{ij}^* of G is a linear combination of three different irreducible characters of G with the multiplicities ± 1. Assume that $t>1$. Since $\psi_{ik}-\psi_{jk} =\phi_{ik}-\phi_{jk}$, the decomposition of ψ_{ik}^* takes the form $\epsilon_k\Phi_{ik}-\epsilon_k'\Xi_k-\epsilon_k''\mathrm{H}_k$. Hence $(\phi_{ik}-\phi_{jk})^*=\psi_{ik}^*-\psi_{jk}^*=\epsilon_k(\Phi_{ik}-\Phi_{jk})$. Suppose $\Phi_{ik}=\Phi_{jl}$. If $k=l$, we must have $i=j$. If $k\neq l$, we take $r\neq i$ and $u\neq j$. Then $\epsilon_k(\phi_{ik}-\phi_{rk})^*-\epsilon_j(\phi_{jl}-\phi_{ul})^* =\Phi_{ul}-\Phi_{rk}$. This is impossible since the right hand side is of norm 4. Here by a norm we shall mean the average of the absolute value squared on G.

Hence $\Phi_{ik}=\Phi_{jl}$ implies $i=j$ and $k=l$. $(\theta_i-\theta_j)^*$ is a difference of two irreducible characters: $(\theta_i-\theta_j)^*=\epsilon(\Theta_i-\Theta_j)$ $(\epsilon=\pm 1)$. If Θ_i coincides with Φ_{lk} (or Ξ_k or H_k), then $\epsilon_k\psi_{lk}^*-\epsilon(\theta_i-\theta_j)^*=\pm\Xi_k\pm\mathrm{H}_k+\Theta_j$. Since the right side is of norm 5, Θ_j must appear in ψ_{lk}^*. ψ_{lk}^* contains 3 characters and j is arbitrary. Hence $s\leqq 3$. Since $m\equiv 1 \pmod{2h}$, we get $q-\delta\leqq 3$ or $q\leqq 4$. We have assumed that $q>2$. Hence $q=4$, $\delta=1$ and $s=3$. Thus $\psi_{lk}^*=\pm\Theta_1\pm\Theta_2\pm\Theta_3$. This is however impossible since $\mathfrak{D}_g\Theta_i=\mathfrak{D}_g\Theta_j$ and $\psi_{lk}^*=0$ on the identity ($\pm 1\pm 1\pm 1=0$ is an impossible relation). Hence Θ_i is different from Φ_{jk}, Ξ_k and H_k. This conclusion is valid even if $t=1$. If Θ_i' is the restriction of Θ_i to H, $\Theta_i'-\epsilon\theta_i$ is orthogonal to all the characters in (*) except possibly the last type. This may be proved for instance as $\langle\Theta_i'-\epsilon\theta_i,\ \theta_j-\theta_k\rangle_H=\langle\Theta_i',\theta_j-\theta_k\rangle_H-\epsilon\langle\theta_i,\theta_j-\theta_k\rangle_H$ $=\langle\Theta_i,\ (\theta_j-\theta_k)^*\rangle_G-\epsilon\langle\theta_i,\ \theta_j-\theta_k\rangle_H=0$. Here $\langle\ ,\ \rangle_G$ means the summation on G divided by the group order and the second equality is a consequence of the reciprocity law of Frobenius. Hence for an element $\rho\sigma$ $(1\neq\rho\in U,\ \sigma\in Z)$ we have $\Theta_i(\rho\sigma)=\epsilon\theta_i(\rho\sigma)$. If $t>1$, the Φ_{ik} are mutually distinct characters. If $\Phi_{ij}=\Xi_k$ (or H_k), $\pm\psi_{ij}^*\pm\psi_{uk}^*$ with suitable coefficients ± 1 is a sum of 4 characters (counting the multiplicity) and of norm 6. Hence either H_k or Φ_{uk} must appear in ψ_{ij}^*. If Φ_{uk} does not appear, $\pm\psi_{rj}^*\pm\psi_{uk}^*$ $(r\neq i)$ is of norm 6 and a sum of 4 distinct characters. Hence Φ_{uk} is contained in ψ_{ij}^* and we have $\pm\psi_{ij}^*=\Phi_{ij}-\epsilon\Phi_{1k}-\epsilon\Phi_{2k}$. This gives however an impossible relation on the identity. Hence if $t>1$, the Φ_{ij} are not equal to Ξ_k nor H_k.

In general suppose that an irreducible character X of G is contained in three characters $\psi_{i1}^*, \psi_{i2}^*, \psi_{i3}^*$. To simplify the notation we drop the first suffix and assume that $\pm\psi_1^*=X+\cdots$, $\pm\psi_2^*=X+\cdots$ and $\pm\psi_3^*=X+\cdots$. Since $\pm\psi_1^*\mp\psi_2^*$ is of norm 6 we see that ψ_1^* and ψ_2^* contain one more common character Y. We may write $\pm\psi_1^*=X+Y-Z$ and $\pm\psi_2^*=X-Y\pm W$. Similarly ψ_1^* and ψ_3^* have one more common character. If $\pm\psi_3^*=X\pm Y\pm W'$, we get a contradiction considering the norm of $\psi_1^*-\psi_2^*$ or $\psi_2^*-\psi_3^*$. Hence $\pm\psi_3^*=X+Z\pm W'$. Considering common characters of ψ_2^* and ψ_3^* we conclude finally $\pm\psi_2^*=X-Y+W$ and $\pm\psi_3^*=X+Z-W$. ψ_1^*, ψ_2^* and ψ_3^* vanish on the identity. Hence the degree of X is 0; a contradiction. Hence no character X appears in 3 different ψ_k^*. Suppose ψ_1^* and ψ_2^* have a common character. Then as before we may write $\pm\psi_1^*=X+Y-Z$ and $\pm\psi_2^*=X-Y\pm W$. Then Z (or W) does not appear in any other ψ_k^*. If so, ψ_k^* and ψ_1^* would have one more common character which is either X or Y. In any case X or Y appears 3 times, which is impossible. From the consideration above we conclude that each ψ_k^* contains a character X_k which does not appear in any other ψ_l^* $(l\neq k)$. If $t>1$, Φ_{ik} is such a character. If $t=1$, we may define $\Phi_{ik}=X_k$. Hence Φ_{ik} is not identical with Ξ_j nor H_j.

If Φ_{ik}' is the restriction of Φ_{ik} to H, then $\Phi_{ik}'-\epsilon_k\phi_{ik}$ is orthogonal to all the first three types of characters in (*). This may be proved exactly as in the case of Θ. Hence $\Phi_{ik}(\rho\sigma)=\epsilon_k\phi_{ik}(\rho\sigma)=\epsilon_k\omega_k(\sigma)\phi_i(\rho)$ for $1\neq\rho\in U$ and $\sigma\in Z$.

As for the value of Ξ_k or H_k on $\rho\sigma$ we can prove similarly that $\Xi_k(\rho\sigma)$

$=\epsilon_k'\omega_k(\sigma)$ and $H_k(\rho\sigma)=\epsilon_k''\omega_k(\sigma)$ if Ξ_k (or H_k) appears only in ψ_k^*, and $\Xi_k(\rho\sigma)=\epsilon_k'\omega_k(\sigma)+\epsilon_l'\omega_l(\sigma)$ and $H_k(\rho\sigma)=\epsilon_k''\omega_k(\sigma)+\epsilon_l''\omega_l(\sigma)$ if Ξ_k is contained in ψ_k^* and ψ_l^*. If ξ_1 is the principal character of H, ψ_{i1}^* contains the principal character of G. This may be shown as $\langle\psi_{i1}^*, 1\rangle_G=\langle\psi_{i1}, \xi_1\rangle_H=-1$. If $t>1$, $\Phi_{i1}(\rho\sigma)=\epsilon_1\omega_1(\sigma)\phi_i(\rho)\neq 1$ for some $\sigma\in Z$ and $1\neq\rho\in U$ and Φ_{i1} is not the principal character. Hence we may take Ξ_1 as the principal character. Then Ξ_1 appears only once and so does H_1. Hence $H_1(\rho\sigma)=\epsilon_1''$ $(\rho\neq 1, \rho\in U, \sigma\in Z)$. If $k>1$, the values $\Xi_k(\rho\sigma)$ and $H_k(\rho\sigma)$ are linear combinations of nonprincipal characters of Z as functions of σ.

We consider the coefficient $A(\sigma)$ of $\langle\rho\sigma\rangle$ $(\rho\neq 1, \rho\in U$ and $\sigma\in Z)$ in the expansion of $\langle\tau\rangle^2$ in the group ring (cf. [1]). Here $\langle\pi\rangle$ is the class of G containing the element π. This coefficient is expressed as

$$A(\sigma) = (g/n^2)\sum_\mu X_\mu(\tau)^2X_\mu(\rho\sigma)/f_\mu,$$

where $g=[G:e]$, n is the order of the centralizer of τ, f_μ is the degree of the irreducible character X_μ and the summation ranges over all the irreducible characters of G. On the other hand $A(\sigma)$ is the number of pairs of conjugate elements τ' and τ'' of τ such that $\tau'\tau''=\rho\sigma$. This number of pairs is equal to the number of conjugate elements of τ which transform $\rho\sigma$ into its inverse. Hence $A(\sigma)=m(q-\delta)$ if $\sigma=1$, and $=0$ if $\sigma\neq 1$. We shall compute the summation over the characters using the values so far obtained. If $X_\mu\neq\Theta, \Phi$, H and Ξ, the restriction of X_μ to H is orthogonal to all the characters (*) except the last ones. Hence $X_\mu(\rho\sigma)=0$. Consider all the Θ's. $\Theta_i-\Theta_j=\epsilon(\theta_i-\theta_j)^*$ vanishes on the identity and on $\langle\tau\rangle$. Hence the degree of Θ_i and the value $\Theta_i(\tau)$ are independent of i. Hence the summation over the Θ's is equal to $(\Theta_i(\tau)^2/(\text{degree of }\Theta_i))\sum_i\Theta_i(\rho\sigma)$. The value $\Theta_i(\rho\sigma)=\epsilon\theta_i(\rho\sigma)$ and $\sum_i\theta_i(\rho\sigma)=0$ if $\rho\neq 1$. Hence the contribution of the Θ's to $A(\sigma)$ is zero. Since $\Phi_{ik}(1)$ and $\Phi_{ik}(\tau)$ are independent of i, the contribution of the Φ_{ik}'s with fixed k to the formula of $A(\sigma)$ is $(\Phi_{ik}(\tau)^2/\Phi_{ik}(1))\sum_i\Phi_{ik}(\rho\sigma)$. Now $\Phi_{ik}(\rho\sigma)=\epsilon_k\omega_k(\sigma)\phi_i(\rho)$ and $\sum_i\phi_i(\rho)=-1$ if $\rho\neq 1$. The contribution is therefore $(-\epsilon_k\Phi_{ik}(\tau)^2/\Phi_{ik}(1))\omega_k(\sigma)$. Assume that $t>1$. Then as shown before we may assume Ξ_1 is the principal character of G. Hence we have

$$A(\sigma) = (g/n^2)\left\{1 + (H_1(\tau)^2/H_1(1))\epsilon_1'' - (\Phi_{i1}(\tau)^2/\Phi_{i1}(1))\epsilon_1 + \sum_{k>1}B_k\omega_k(\sigma)\right\},$$

where B_k is constant as a function on Z. Summing over Z we get

$$m(q-\delta) = (g/n^2)h\{1 + (H_1(\tau)^2/H_1(1))\epsilon_1'' - (\Phi_{i1}(\tau)^2/\Phi_{i1}(1))\epsilon_1\}.$$

Let $H_1(\tau)=a$ and $H_1(1)=f$. Since $\epsilon_1\Phi_{i1}-1-\epsilon_1''H_1=0$ on the identity and on τ, we get $\epsilon_1=\epsilon_1''$ (call it ϵ), $\Phi_{i1}(1)=f+\epsilon$ and $\Phi_{i1}(\tau)=a+\epsilon$. Hence

$$m(q-\delta) = (gh/n^2)(1 + (a^2/f)\epsilon - \epsilon(a+\epsilon)^2/(f+\epsilon)),$$

$$g = n^2m(q-\delta)f(f+\epsilon)/h(f-a)^2.$$

This is the formula we are looking for.

If $t=1$, we have the possibility that Φ_{11} is the principal character and hence a slight modification of the argument is necessary. Nevertheless we have exactly three characters involving the principal character of U as the value on $\langle\rho\sigma\rangle$ and the final formula is still valid.

Assuming $t>1$ and $s\geqq 1$, we consider the decomposition of the induced character ψ^* of $\psi=\theta_i-\sum_k \phi_{jk}$. The norm of ψ^* is $1+h$. Let a_μ be the multiplicity of Θ_μ in ψ^*. $\psi^*-(\theta_i-\theta_l)^*$ has the same norm $1+h$. Hence $a_i^2+a_l^2=(a_i-\epsilon)^2+(a_l+\epsilon)^2$ or $a_l=a_i-\epsilon$. Hence $\psi^*=\epsilon\Theta_i+a\sum_\mu \Theta_\mu+\cdots$. Thus the contribution of the Θ's to the norm is $(a+\epsilon)^2+(s-1)a^2\geqq 1$. For a fixed l let b_μ be the multiplicity of $\Phi_{\mu l}$ in ψ^*. We use the same argument as above. $\psi^*+(\phi_{jl}-\phi_{\mu l})^*$ has the same norm $1+h$. Hence $b_j^2+b_\mu^2=(b_j+\epsilon_l)^2-(b_\mu-\epsilon_l)^2$ and as before $b_\mu=b_j+\epsilon_l$. Therefore $\psi^*=\cdots+b\sum_\mu \Phi_{\mu l}-\epsilon_l\Phi_{jl}+\cdots$. Again the contribution of the $\Phi_{\mu l}$'s to the norm is $\geqq 1$. Since the norm of ψ^* is exactly $1+h$, we see that the contributions are all 1. Hence ψ^* contains Θ_i, one $\Phi_{\mu l}$ $(l=1, 2, \cdots, h)$ and nothing else. In particular ψ^* does not contain H_1. If H_1' is the restriction of H_1 to H, $\mathrm{H}_1'-\epsilon\eta_1$ is orthogonal to all the characters in (*). This implies that $\mathrm{H}_1(\pi)=\epsilon$ for all $\pi\neq 1$ of W. Hence the degree of H_1 is congruent to ϵ modulo $m(q-\delta)$. The same assertion is also true even if $s=0$, since in this case $m=1$ and $\mathrm{H}_1(\rho)=\epsilon$ for all $\rho\neq 1$ of U.

We have thus shown the validity of the following proposition.

PROPOSITION 6. *Let G be a finite group of even order satisfying conditions* (A), (B) *and* (C). *Let τ be an involution contained in H (notations being the same as in Proposition* 5). *Denote by m the index $[W: U]$ and by n the order of the centralizer of τ in G. We conclude that there exist irreducible characters $X, Y_1, \cdots, Y_t$ $(2t+1=q-\delta)$ such that $X(\sigma)+\epsilon=Y_i(\sigma)$ $(\epsilon=\pm 1; i=1, 2, \cdots, t)$ for all elements σ with order relatively prime to $q-\delta$, and that the order g of G has the form*

$$g = n^2m(q-\delta)f(f+\epsilon)/h(f-a)^2$$

where f is the degree of the character X and $a=X(\tau)$. The degree f satisfies the congruence $f\equiv\epsilon$ (mod $q-\delta$). If $t>1$, then $X(\sigma)=\epsilon$ on $\sigma\neq 1$ of W and hence $f\equiv\epsilon$ (mod $m(q-\delta)$).

3. We shall return to the proof of our main theorem. If the conclusion of the main theorem is not true we have shown the existence of a group G satisfying the conditions of Proposition 4. Our group G satisfies the first assumption (A) of the preceding section. Note the change in notation. Since every involution in G has an abelian centralizer which is a direct product of A and a 2-Sylow subgroup, the second assumption (B) is also satisfied. G satisfies the assumption (C) trivially. Hence by Proposition 6 G has two irreducible characters X and $Y=Y_1$ of degrees f and $f+\epsilon$ $(\epsilon=\pm 1)$. A 2-Sylow subgroup S of G is of order exactly q and consists of $q-1$ involutions conjugate to the involution τ. The orthogonality relation yields that $f+(q-1)a\equiv 0$ (mod q) or

$f \equiv a \pmod{q}$ where $a = X(\tau)$. The order n of the centralizer of τ is qh. Since q divides g and the index m is odd, we conclude that q is a divisor of $f(f+\epsilon)$ from the order formula in Proposition 6. X and Y satisfy the relation that $X(\sigma)+\epsilon = Y(\sigma)$ for any element σ of order relatively prime to $q-\delta$. In particular for $\sigma = \tau$ we have $Y(\tau) = a+\epsilon$. Now q is a divisor of $f(f+\epsilon)$ and so q divides either f or $f+\epsilon$. Since q is the highest power of 2 dividing the order g of G either X or Y is of highest kind and hence (cf. [2]) vanishes on τ. Therefore we have either $a=0$ or $a=-\epsilon$. Let $f = qf'$ or $f+\epsilon = qf'$ accordingly. Then f' is an odd integer and we have

$$g = qhm(q - \delta)(f'q \pm \epsilon)/f'.$$

Since $hm(q-\delta)$ is the order of a subgroup, $q(f'q \pm \epsilon)/f'$ is an integer as being the index. Since f' is odd, f' is a divisor of $f'q \pm \epsilon$. Hence f' is 1 as a divisor of ± 1. This means that either $f=q$ or $f+\epsilon = q$. If $q-\delta > 3$, then from the last assertion of Proposition 6 we have $f \equiv \epsilon \pmod{m(q-\delta)}$. Hence $q-\epsilon$ or $q-2\epsilon$ is a multiple of $m(q-\delta)$. If $m>1$, we have $q+2 \geqq 3(q-1)$ or $5 \geqq 2q$. This is impossible. Hence $m=1$ and $g = qh(q-\delta)(q \pm \epsilon)$. Since g/h is a multiple of $q(q^2-1)$ we conclude that $q \pm \epsilon = q+\delta$ and $g = hq(q^2-1)$. Since M was a maximal subgroup of order $hq(q^2-1)$ we get the contradiction $G = M$.

Assume that $q-\delta \leqq 3$. Then $q=4$ and $\delta = 1$. h is a divisor >1 of $q+\delta = 5$. Hence $h=5$. We have $m \equiv 1 \pmod{10}$. The formula for the order now reads $g = 4 \cdot 5 \cdot m \cdot 3\ (4 \pm \epsilon)$. In the notations of the §2, $3m$ is the order of W and $2(4 \pm \epsilon)$ is the index of H. Hence by Sylow's theorem $2(4 \pm \epsilon) \equiv 1 \pmod{3m}$ which implies $9 \geqq 3m$. Since $m \equiv 1 \pmod{10}$ we have $m=1$. We get the same contradiction as in the case of $q-\delta > 3$.

We have shown the impossibility of a finite group satisfying the conditions of Proposition 4. This completes the proof of our theorem stated in the introduction.

4. As an application of our main theorem we prove the following theorem.

THEOREM. *Let G be a finite nonsolvable group. We assume that the Sylow subgroups belonging to odd primes are cyclic and the 2-Sylow subgroups are abelian. If we assume moreover that 2-Sylow subgroups are independent, i.e. each two different 2-Sylow subgroups contain only the identity in common, then G contains a normal subgroup G_0 isomorphic with $LF(2, 2^\mu)$ and the factor group G/G_0 is a solvable group of odd order.*

Proof. Use an induction on the order of G. Assume that the commutator subgroup G' of G is a proper subgroup. Then we may apply the inductive hypothesis to G'. G' contains therefore a normal subgroup G_0 isomorphic with $LF(2, 2^\mu)$ and the factor group G'/G_0 is solvable. G_0 is a characteristic subgroup of G' and hence a normal subgroup of G. The factor group G/G_0 is solvable. We want to show the order of G/G_0 is odd. By way of contradiction we assume that the order of G/G_0 is even. We may assume that G/G_0 is of order 2. Let S be a 2-Sylow subgroup of G_0. By a theorem of Sylow S is contained in a 2-Sylow subgroup T of G. Let N be the normalizer of S in G. $N\cap G_0$ is the normalizer of S in G_0. From the structure of $G_0\cong LF(2, 2^\mu)$, $N\cap G_0/S$ is a cyclic group of odd order n. T/S is the only 2-Sylow subgroup of N/S since otherwise S is a part of two different 2-Sylow subgroups. Hence T is a normal subgroup of N. Therefore N/S is the direct product of T/S and $N\cap G_0/S$, and hence N/S is a cyclic group. Hence N contains an element σ of order a multiple of $2n$; σ^2 is an element of $N\cap G_0$ and of order a multiple of n. Hence the order of σ is exactly $2n$. It follows easily that σ^n is an element of N which commutes with every element of N, since $\sigma^n\in T$ and T is abelian by assumption. From the structure of G_0 there is another 2-Sylow subgroup S' of G_0 such that $\sigma^2S'\sigma^{-2}=S'$. Since S and S' are the only 2-Sylow subgroups of G_0 which commute with σ^2 we conclude that $\sigma^nS'\sigma^{-n}=S'$. This means σ^n and S' generate a 2-group T'. Since $T'\cap G_0=S'$, T' is different from T and both T and T' contain σ^n. This is a contradiction to the assumption of independency of 2-Sylow subgroups. Thus if the commutator subgroup G' is a proper subgroup of G our conclusion is valid.

Assume that the commutator subgroup of G coincides with G. Let H be the normalizer of a 2-Sylow subgroup S. By a theorem of Schur (cf. [9, p. 125]) H contains a subgroup K such that $KS=H$, $K\cap S=e$ and $K\cong H/S$. By assumption all the Sylow subgroups of K are cyclic.

We shall first prove that the involutions of G form a single conjugate class. A 2-Sylow subgroup S is not a normal subgroup. If so, a theorem of Burnside [4, §243] shows the solvability of G/S and hence of G. This contracts the assumption of nonsolvability of G. We may therefore take another 2-Sylow subgroup S'. If $\tau\in S$ and $\tau'\in S'$ are two involutions the subgroup $\{\tau, \tau'\}$ generated by τ and τ' is a dihedral group. Since 2-Sylow subgroups are independent, τ and τ' are conjugate in $\{\tau, \tau'\}$. If τ'' is another involution of S, τ'' is conjugate to τ' and hence τ and τ'' are conjugate to each other. Thus the involutions of G form a single conjugate class.

Consider the centralizer Z of τ in G. If $\sigma\in Z$, $\sigma S\sigma^{-1}\ni\sigma\tau\sigma^{-1}=\tau$. Hence $\sigma S\sigma^{-1}$ and S have τ in common. Hence $\sigma S\sigma^{-1}=S$; namely σ is an element of H. We have thus shown $Z\subseteq H$.

Let $U=Z\cap K$. We want to show that if two elements σ and ρ of U are conjugate in G then they are conjugate in H. Suppose that there is an element $\pi\in G$ such that $\pi\sigma\pi^{-1}=\rho$. The centralizer V of σ is mapped on the centralizer W of ρ: $\pi V\pi^{-1}=W$. Hence $\pi\tau\pi^{-1}$ and τ are two involutions of W. If a 2-Sylow subgroup T of W is normal, then T is a part of S since $T\cap S\ni\tau$. Since $\pi\tau\pi^{-1}$ is also in T, we see that $\pi S\pi^{-1}\cap S\neq e$ and hence $\pi S\pi^{-1}=S$ by our assumption. This means that $\pi\in H$. If W contains more than one 2-Sylow subgroup the involutions of W form a single conjugate class. Hence there is an element ν of W such that $\nu\pi\tau\pi^{-1}\nu^{-1}=\tau$. This implies that $\nu\pi\in Z\subseteq H$. Since $\nu\in W$, $\nu\pi\sigma\pi^{-1}\nu^{-1}=\nu\rho\nu^{-1}=\rho$. Thus in any case σ and ρ are conjugate in H.

Let K' be a commutator subgroup of K. If $U \not\subseteq K'$, we may take a prime divisor p of the index $[U: U \cap K']$ and consider a p-Sylow subgroup P of U. P is contained in a p-Sylow subgroup Q of G. Q is by assumption a cyclic group. If σ and ρ are two different elements of P, σ and ρ are not conjugate in H since p is a divisor of $[K: K']$. Hence σ and ρ are not conjugate in G. This implies that Q is in the center of its normalizer. Burnside's theorem (loc. cit.) may be applied to show the existence of a proper normal subgroup with cyclic factor group. This contradicts our assumption of perfectness of G. Hence U is a subgroup of K'. From the structure of K (cf. [8]) it follows that U is a normal subgroup of K and is cyclic. Hence $Z = US$ is a normal subgroup of H. Since every involution of S is conjugate to τ in H, Z is the centralizer of any involution of S. Each element of U induces an automorphism of the abelian group S which leaves every involution invariant. Since U has an odd order every element of U induces the trivial automorphism of S. Hence Z is the direct product of U and S, and hence Z is abelian. Z is the centralizer of an involution τ and every involution of G is conjugate to τ. Our main theorem of this paper may be applied here. Since G is perfect we conclude that G is isomorphic with $LF(2, 2^{\mu})$. This completes the inductive proof of our theorem.

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